Equilibrium Asset Pricing with Frictions

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Outline

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1. Equilibrium Asset Pricing with Complete Markets
2. Incomplete Markets
3. A Framework with Many Nested Models
4. The Dark Side of Circuit Breakers
   - Overview
   - The Model
   - Impact of Circuit Breakers
   - Conclusion

Contingent Claims Equilibrium

- $(\Omega, \mathscr{F}, P)$, $\mathscr{T} = \{0, 1, \cdots, T\}$, and $\mathbb{F}$.
- Full set of contingent claims: in zero net supply; can be traded only at date 0.
- I agents: $U_i : \mathscr{L}^+ \to \mathbb{R}$ strictly increasing and continuous.
- Endowment for agent i: $e_i \in \mathscr{L}^+$

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Definition

A contingent claims (CC) equilibrium is a price process $\psi \in \mathscr{L}$ and a vector of consumption plans $(c_1, .., c_I) \in \mathscr{L}_I$ such that

1. *(optimization) c_i is optimal for agent i;*
2. *(market-clearing) $\sum_{i=1}^{I} c_i = \sum_{i=1}^{I} e_i$.*

Pareto Optimality

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A consumption allocation is a vector $(c^1, .., c^I)$ *of consumption plans.*

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A feasible consumption allocation $(c^1, .., c^I)$ *is Pareto optimal iff there does not exist a feasible allocation* $(\widehat{c}^1, .., \widehat{c}^I)$ *such that* $U_i(\widehat{c}_i) \ge U_i(c_i)$ *for all* i*, and* $U_i(\widehat{c}_i) > U_i(c_i)$ *for at least one* i.

PARETO OPTIMALITY

FIRST WELFARE THEOREM

The CC equilibrium consumption allocation is Pareto optimal.

Representative Agent

Representative agent problem ($\mathscr{R}$)

For a set of weights λ_i, we define

$$U(c) = \max_{c_1,..,c_I} \sum_{i=1}^{I} \frac{1}{\lambda_i} U_i(c_i)$$

$$c_i \in \mathscr{L}^+, \quad \sum_{i=1}^{I} c_i \leq c$$

Representative Agent

Theorem

There exist weights λ_i such that the equilibrium consumption allocation solves the problem $\mathscr{R}$ for the aggregate consumption $c = e$. Moreover, the equilibrium price process ψ and the consumption plan $c = e$ is an equilibrium for the representative agent economy.

Representative Agent

Proposition

Suppose that for all i, U_i is a time-additive expected utility, $U_i(c_i) = E_0 \sum_{t=0}^{T} u_{i,t}(c_{i,t})$. Then U is a time-additive expected utility,

$$U(c) = E_0 \sum_{t=0}^{T} u_t(c_t).$$

Moreover,

$$u_t(c_t) = \max_{c_{1,t},..,c_{I,t}} \sum_{i=1}^{I} \frac{1}{\lambda_i} u_{i,t}(c_{i,t}),$$

$$c_{i,t} \in \mathscr{L}^+, \quad \sum_{i=1}^{I} c_{i,t} \le c_t.$$

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- Agent i's problem $\mathscr{P}_i$ is

$$\max_{c_i} U_i(c_i)$$

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- A consumption plan c_i is optimal iff it solves $\mathscr{P}_i$. A trading strategy θ_i is optimal iff it finances $c_i^* - e_i - \overline{\theta}_{i,0} S_0$.

Securities Market Equilibrium

Definition

A securities market (SM) equilibrium is a price process $S \in \mathscr{L}^N$ and a vector of trading strategies $(\theta_1, .., \theta_I) \in \mathscr{L}^{NI}$ such that

1. *(optimization) θ_i is optimal for agent i*
2. *(market-clearing)*
$$\sum_{i=1}^{I} \theta_i = x, \quad \sum_{i=1}^{I} c_i = \sum_{i=1}^{I} e_i + x\delta.$$

Securities Market Equilibrium

Example

- *Consider an economy with two agents. Both have time separable preferences of the form*

$$U_i = E_0 \left[\sum_{t=0}^{T} \frac{1}{1-\gamma_i} [(c_{i,t})^{1-\gamma_i} - 1] \right]$$

where $\gamma_1 = 1$ *and* $\gamma_2 = 1/2$.

- *There exists a complete set of contingent claims in zero net supply and also a single share of long-lived asset, stock, with a dividend process* δ_t. *Each agent is endowed with half a share at time zero.*
- *Characterize the stock price process in this economy.*

Model Setup in Continuous Time

- Consider a probability space $(\Omega, \mathscr{F}, P)$, a time interval $\mathscr{T} = [0, T]$, a Brownian motion $Z = (Z_1, .., Z_d)$ on $(\Omega, \mathscr{F}, P)$, and the standard filtration $\mathbb{F}$ of Z.

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- I agents:

$$U_i(c_i, C_{i,T}) = E\left[\int_0^T u_{i,t}(c_{i,t})\,dt + U_{i,T}(C_{i,T})\right]$$

  $u_{i,t}$ and $U_{i,T}$ strictly increasing and concave.

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- Endowment: Agent i receives an endowment of the consumption good at a rate $e_i \in \mathscr{L}^1$. He also receives an endowment $\overline{\theta}_{i,0}$ of the securities at time 0; $\sum_{i=1}^I \overline{\theta}_{i,0} = x$.

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- Security price processes:

$$dS_t = I_{S_t}\overline{\mu}_t dt + I_{S_t}\overline{\sigma}_t dZ_t,$$

  where $\mu \in (\mathscr{L}^1)^N$ and $\sigma \in (\mathscr{L}^2)^{N\times d}$.

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- We assume that trading strategies are in $\mathscr{L}(S)$ and are such that the stochastic integral $\int_0^t \theta_s d(S_s/B_s)$ is a martingale under Q.

Equilibrium in Continuous Time

Individual agent problem: $\mathscr{P}_i$

$$\max_{c_i, C_{i,T}} U(c_i, C_{i,T})$$

$$(c_i, C_{i,T}) \in \mathscr{C}_i,$$

where $\mathscr{C}_i$ the set of feasible cash flows for agent i. A consumption plan $(c_i, C_{i,T})$ is optimal iff it solves $\mathscr{P}_i$.

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1. *(optimization) (c_i, θ_i) is optimal for agent i*
2. *(market-clearing)*

$$\sum_{i=1}^{I} \theta_i = x, \quad x\delta + \sum_{i=1}^{I} (e_i - c_i) = 0.$$

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  - ↪ At time $t = 0$, agent 2 is endowed with A units of the bond, priced at 1.
  - ↪ Agent 1 is endowed with one share of the stock, is short A shares of the bond.

Individual optimization

- Consider the constrained agent.
- Market price of risk in fictitious market $\eta^{(\lambda)}$, interest rate $r^{(\lambda)} = r + \delta(\lambda_t)$.
- Since the constrained agent can freely access the bond market,

$$r_t^{(\lambda)} = r_t$$

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- State-price density of the constrained agent

$$d\pi_t^{(c)} = -r_t \pi_t^{(c)}\, dt \quad \Rightarrow \quad \pi_t^{(c)} = B_t^{-1}$$

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- π_t is state-price density of the unconstrained agent, η is market price of risk

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- Consumption-sharing rule

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- Solution: $c_{1,t} = F(\xi_t, \delta_t)$

Market clearing

- Characterize dynamics of ξ_t using market clearing conditions.

$$\frac{d\xi_t}{\xi_t} = \frac{d\pi_t}{\pi_t} + \frac{dB_t}{B_t} = -r_t\,dt - \eta_t\,dZ_t + r_t\,dt = -\eta_t\,dZ_t$$

Then

$$\frac{d\xi_t}{\xi_t} = \text{stoch}\left(\frac{u_1''(c_{1,t})}{u_1'(c_{1,t})}\,dc_{1,t}\right) = \frac{u_1''(F(\xi_t,\delta_t))}{u_1'(F(\xi_t,\delta_t))}\,\sigma\delta_t\,dZ_t$$

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$$dc_{1,t} = d\delta_t - dc_{2,t}$$

and

$$\text{stoch}(dc_{2,t}) = 0$$

- Given initial value ξ_0, completely characterize equilibrium allocations.
- Then compute prices using SPD

$$\pi_t = e^{-\rho t}\frac{u_1'(c_{1,t})}{u_1'(c_{1,0})}$$

Equilibrium

- Initial condition ξ_0 determined by budget constraint of agent 2.
- Optimal consumption policy of the log agent

$$c_{2,0} = W_{2,0} \left(\frac{1 - e^{-\rho T}}{\rho} \right)^{-1}, \quad W_{2,0} = A$$

- Using definition of ξ_t

$$\xi_t = u_1'(c_{1,t}) c_{2,t}$$

- ξ_0 must solve

$$\frac{\xi_0}{u_1'(F(\xi_0, \delta_0))} \frac{1 - e^{-\rho T}}{\rho} = A$$

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- ξ_0 must solve

$$\frac{\xi_0}{u_1'(F(\xi_0,\delta_0))}\frac{1-e^{-\rho T}}{\rho} = A$$

- Limited participation increases market price of risk

$$\eta_t = -\frac{c_{1,t}u_1''(c_{1,t})}{u_1'(c_{1,t})}\frac{\delta_t}{c_{1,t}}\sigma$$

Discussion

- It is common to interpret ξ_t as the stochastic utility weight

$$\sup u_1(c_1) + \xi\, u_2(c_2) \quad \text{s.t. } c_1 + c_2 = \delta$$

- Stochastic Pareto-Negishi weights, ξ_t.

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$$\sup u_1(c_1) + \xi\, u_2(c_2) \quad \text{s.t. } c_1 + c_2 = \delta$$

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- Stochastic Pareto-Negishi weights, ξ_t.
- Solve for ξ_t instead of searching for price processes directly.
- Note: high volatility of ξ_t implies high volatility of SDF.
- Under complete markets, ξ_t is constant.

Logarithmic preferences

- Assume agent 1 also has log utility.

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- At $t = 0$,

$$\eta_0 = (1+\xi_0)\sigma \qquad \nearrow \quad \text{in} \quad \frac{W_{2,0}}{W_{1,0}}$$

Logarithmic preferences

- Risk-free rate

$$\pi_t = e^{-\rho t}\frac{1+\xi_t}{\delta_t}, \quad E_t\left[-\frac{d\pi_t}{\pi_t}\right] = r_t\,dt$$

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LOGARITHMIC PREFERENCES

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- For large enough $\frac{W_{2,0}}{W_{1,0}}$, obtain a high market price of risk and a low risk-free rate.
- Volatility of stock returns is still equal to the volatility of dividend growth.

OUTLINE

Hansen, Huang, Khorrami, Tourre (2018)

- **Starting Point:** Brunnermeier & Sannikov (2016)

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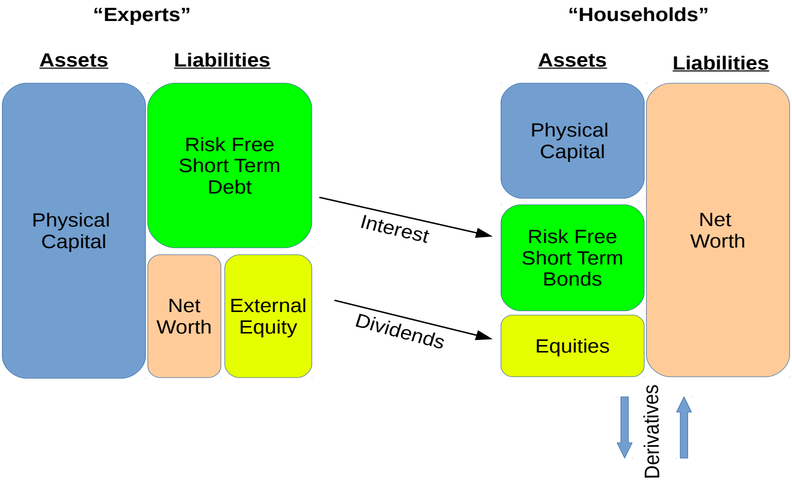
- **Preferences**
  - ↪ Recursive utility
  - ↪ Households and experts potentially different
  - ↪ OLG for technical reasons

“Nesting” Model



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  - ↪ Brunnermeier & Sannikov (2014, 2016)

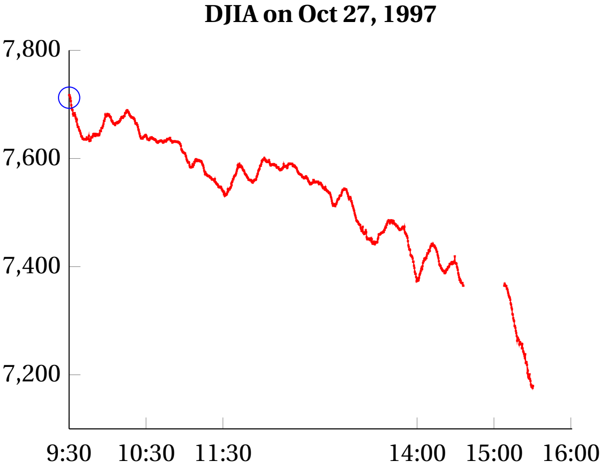
Outline

Circuit Breakers

- What is it?
  - ↪ Trading halt following extreme price movements.
  - ↪ Market-wide CBs; CBs for individual stocks.
  - ↪ First advocated by the Brady Commission following the Black Monday of 1987. Now widely adopted around the world.
- Why?
  - ↪ To reduce excess volatility and improve price efficiency?
  - ↪ To restore orderly trading in the market?
  - ↪ To protect investors?
- What are the consequences?

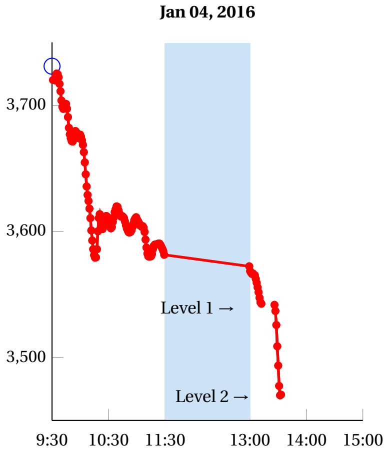
Circuit Breakers: U.S. Experience

- Market-wide CB was triggered only once in the U.S. since 1988.



Circuit Breakers: Chinese Experience

- First implemented on Jan 04, 2016, following the market crash in summer 2015. Abandoned after just 4 days.



Overview of Results

- This paper: A neoclassical benchmark to examine how CBs affect trading and price dynamics.
  - ↪ Abstract away from informational frictions, strategic behavior.
  - ↪ Focus on the basic risk-sharing trading motive.
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- CBs tend to have the following effects:
  1. Price level $\Downarrow$ (price distortion $\Uparrow$)
  2. Volatility: daily price range $\Downarrow$ conditional & realized vol $\Uparrow$
  3. “Magnet effect”: increase “hitting probability” relative to complete markets.
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- Policy implications

- Model is tractable and can be adapted to study dynamic effects of illiquidity in other settings.

Related Literature

- Theory on trading halts:
  - ↪ Greenwald and Stein (1988, 1991)
  - ↪ Subrahmanyam (1994, 1995)
  - ↪ Periodic trading halts: Hong and Wang (2000)
- Runs in financial markets: Diamond and Dybvig (1983), Bernardo and Welch (2004)
- Empirical evidence: Lee, Ready, Seguin (1994), Christie, Corwin, Harris (2002), Goldstein and Kavajecz (2004)

Model Setup

- A continuous-time endowment economy over interval $[0, T]$.
- Aggregate stock: one unit, with terminal dividend D_T.

$$dD_t = \mu D_t dt + \sigma D_t dZ_t, \quad D_0 = 1$$

- Riskless bond: net supply Δ, pays off 1 at time T.

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- Riskless bond: net supply Δ, pays off 1 at time T.
- Two competitive agents: A and B
  - ↪ Endowed with ω and $1 - \omega$ shares of the stock and bond.
  - ↪ Log preferences over terminal wealth

$$u_i(W_T^i) = \ln(W_T^i), \quad i = \{A, B\}$$

- No intermediate consumption ⇒ riskless bond as numeraire.

Heterogeneous Beliefs

- The two agents disagree about the growth rate of dividend.
- Agent A has objective beliefs:

$$\mu^A = \mu$$

- Agent B's belief:

$$\mu_t^B = \mu + \delta_t$$

  - ↪ Constant disagreement: $\delta_t \equiv \delta$
  - ↪ Extrapolative disagreement: $d\delta_t = \nu dZ_t$

- The two agents "agree to disagree."

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- The two agents "agree to disagree."
- Need trading. Heterogeneous risk aversion works similarly.

Heterogeneous Beliefs

- Agent B's probability measure $\mathbb{P}^B$ is equivalent to $\mathbb{P}$.
- Radon-Nikodym derivative of measure $\mathbb{P}^B$ with respect to $\mathbb{P}$:

$$\eta_t = \exp\left(\frac{1}{\sigma}\int_0^t \delta_s dZ_s - \frac{1}{2\sigma^2}\int_0^t \delta_s^2 ds\right)$$

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- Intuition: Think of η_t as likelihood ratio.
  - $\hookrightarrow$ Agent B will be more optimistic than A when $\delta_t > 0$. Then, those paths with high realized values for $\int_0^t \delta_s Z_s$ will be assigned higher probabilities under $\mathbb{P}^B$ than under $\mathbb{P}$.

Circuit Breakers

- The stock market will be closed until T whenever the price of the stock S_t falls below the level $(1-\alpha)S_0$.

$$\tau = \inf\{t \geq 0 : S_t = (1-\alpha)S_0\}$$

  - $\hookrightarrow$ α: circuit breaker limit, $\alpha \in [0,1]$
  - $\hookrightarrow$ S_0: initial stock price — endogenous

- After stock market closure agents are not able to change their stock postions

- Bond market remains open throughout the interval $[0, T]$.

Equilibrium: No Circuit Breakers

- Markets are dynamically complete ⇒ solve for equilibrium via the planner's problem.

$$\max_{\widehat{W}_T^A, \widehat{W}_T^B} \mathbb{E}_0 \left[\lambda \ln\left(\widehat{W}_T^A\right) + (1-\lambda)\eta_T \ln\left(\widehat{W}_T^B\right) \right]$$

subject to resource constraint

$$\widehat{W}_T^A + \widehat{W}_T^B = D_T + \Delta$$

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  subject to resource constraint

$$\widehat{W}_T^A + \widehat{W}_T^B = D_T + \Delta$$

- Stock price when $\Delta \to 0$: wealth-weighted average of the prices under two agents' beliefs

$$\widehat{S}_t = \left(\frac{\widehat{\omega}_t^A}{\widehat{S}_t^A} + \frac{\widehat{\omega}_t^B}{\widehat{S}_t^B} \right)^{-1}$$

  - $\widehat{S}_t^i$: price in a single-agent economy with agent i
  - $\widehat{\omega}_t^i$: agent i's wealth share

Equilibrium: No Circuit Breakers

- Special case: constant disagreement

$$\widehat{S}_t = \left(\frac{\widehat{\omega}_t^A}{\widehat{S}_t^A} + \frac{\widehat{\omega}_t^B}{\widehat{S}_t^B} \right)^{-1}$$

where

$$\widehat{S}_t^A = D_t e^{(\mu - \sigma^2)(T-t)}$$

$$\widehat{S}_t^B = D_t e^{(\mu + \delta - \sigma^2)(T-t)}$$

- Valuation gap: $e^{\delta(T-t)}$

Equilibrium: Circuit Breakers

- Two scenarios:
  - $\hookrightarrow$ CB is not triggered between 0 and T;
  - $\hookrightarrow$ CB is triggered at time $\tau < T$.

- Markets are still dynamically complete over interval $[0, \tau \wedge T]$.

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- Two scenarios:
  - ↪ CB is not triggered between 0 and T;
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- Solution strategy:
  1. Pin down stopping rule τ consistent with a given stopping price $\underline{S}$ through equilibrium conditions upon market closure.
  2. Given stopping time τ, solve for equilibrium allocation at $\tau \wedge T$ via planner's problem.
  3. Compute price at $t \leq \tau \wedge T$ for given τ and $\underline{S}$, $S_t(\tau, \underline{S})$.
  4. Solve for $\underline{S}$ through the fixed point problem,

$$\underline{S} = (1 - \alpha) S_0(\underline{S})$$

Equilibrium: Upon Market Closure

- Suppose agent i has wealth W_τ^i at time $\tau \leq T$.
- Portfolio problem at time τ for competitive agents:

$$V^i(W_\tau^i, \tau) = \max_{\theta_\tau^i, \varphi_\tau^i} \mathbb{E}_\tau^i \left[\ln(\theta_\tau^i D_T + \phi_\tau^i) \right]$$

$$s.t. \quad \theta_\tau^i S_\tau + \phi_\tau^i = W_\tau^i$$

$$W_T^i \geq 0$$

$V^i(W_\tau^i, \tau)$: indirect utility function for agent i at time τ

- Market clearing conditions:

$$\theta_\tau^A + \theta_\tau^B = 1 \quad \text{(stock market)}$$

$$\phi_\tau^A + \phi_\tau^B = \Delta \quad \text{(bond market)}$$

Equilibrium ($\Delta \to 0$ case): Upon Market Closure

- Market closure $\Rightarrow$ inability to rebalance between τ and T
  - $\hookrightarrow$ Illiquidity + log utility $\Rightarrow$ no short or levered position at τ
  - $\hookrightarrow$ Leverage constraint binds for the optimistic agent
    $\Rightarrow$ pessimistic agent becomes the marginal investor.
  - $\hookrightarrow$ Assumption $\Delta \to 0$ to be relaxed later.

EQUILIBRIUM ($\Delta \to 0$ CASE): UPON MARKET CLOSURE

- Market closure $\Rightarrow$ inability to rebalance between τ and T
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PROPOSITION

In the limiting case with $\Delta \to 0$, upon market closure at $\tau < T$, both agents will hold all of their wealth in the stock with no bonds. The market clearing price is

$$S_\tau = \min\{\widehat{S}^A_\tau, \widehat{S}^B_\tau\}$$

- Stopping rule τ is expressed in closed form as a function of state variables.

CHARACTERIZING THE STOPPING TIME τ

LEMMA

Take the stopping price $\underline{S}$ as given. Define a stopping time

$$\tau = \inf\{t \geq 0 : D_t = \underline{D}(t, \delta_t, \underline{S})\}.$$

Then the circuit breaker is triggered at time τ when $\tau \leq T$.

Characterizing the stopping time τ

Lemma

Take the stopping price $\underline{S}$ as given. Define a stopping time

$$\tau = \inf\{t \geq 0 : D_t = \underline{D}(t, \delta_t, \underline{S})\}.$$

Then the circuit breaker is triggered at time τ when $\tau \leq T$.

- We have managed to characterize a stopping time that is based on the endogenous stock price S_t as one that is based on the exogenous processes of D_t and δ_t.

Equilibrium: Before Market Closure

- Solve for optimal allocation at $\tau \wedge T$ through the planner problem, using the indirect utilities upon market closure:

$$\max_{W^A_{\tau\wedge T}, W^B_{\tau\wedge T}} \mathbb{E}_0 \left[\lambda V^A(W^A_{\tau\wedge T}, \tau \wedge T) + (1-\lambda)\eta_T V^B(W^B_{\tau\wedge T}, \tau\wedge T) \right]$$

subject to

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- The price of the stock at time $t \leq \tau \wedge T$:

$$S_t = \mathbb{E}_t\left[\frac{\pi^A_{\tau\wedge T}}{\pi^A_t} S_{\tau\wedge T}\right] = \left(\omega^A_t \mathbb{E}_t[S^{-1}_{\tau\wedge T}] + \omega^B_t \mathbb{E}^B_t[S^{-1}_{\tau\wedge T}]\right)^{-1}$$

Equilibrium: $\underline{S}$

- Fixed point problem

$$\underline{S} = (1-\alpha) S_0(\tau(\underline{S}), \underline{S}) \qquad (*)$$

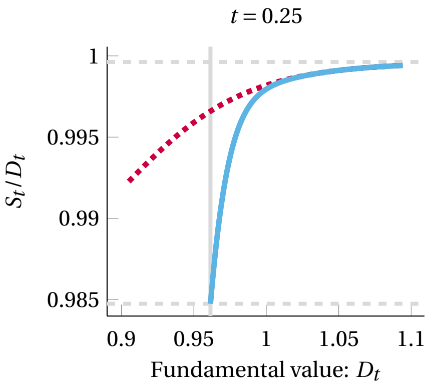
Proposition

There is a unique solution to $(*)$ for any $\alpha \in [0,1]$.

Special Case: Constant Disagreement

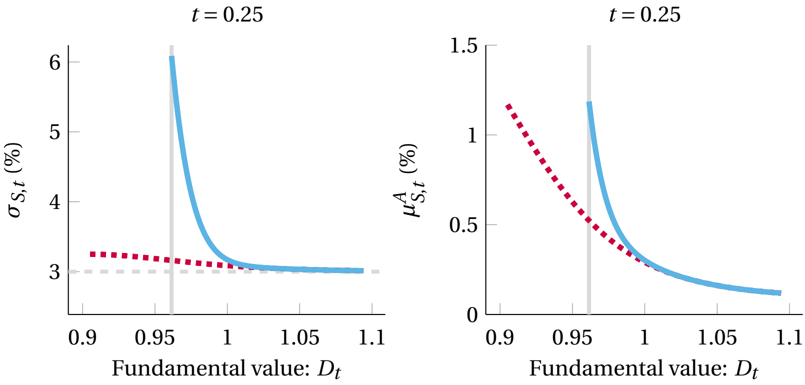
- Calibration:
  - $T = 1$
  - $\mu = 10\%/250$
  - $\sigma = 3\%$
  - $\alpha = 5\%$
  - $\delta = -2\%$ Agent B is pessimistic.
  - $\omega = 90\%$ Most wealth initially owned by rational agent.

PRICE AND AGENT A'S PORTFOLIO HOLDINGS



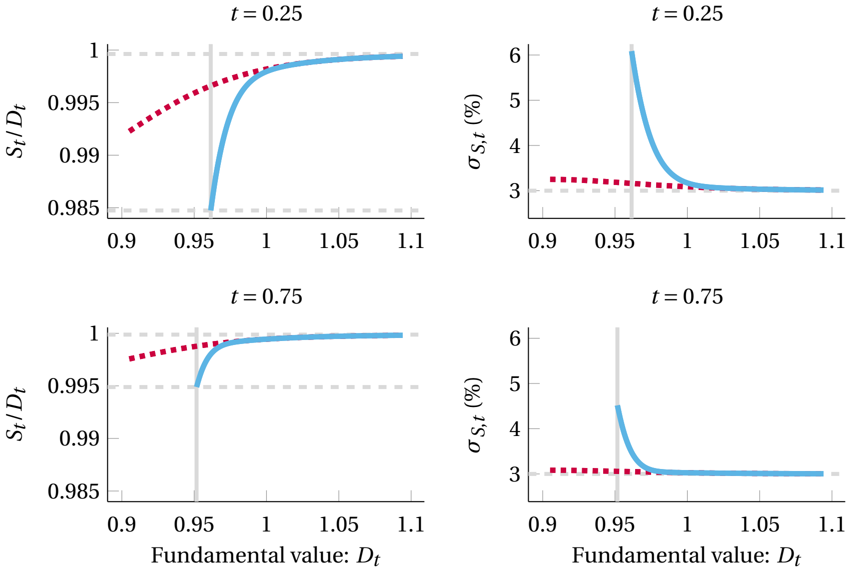
Dotted line – complete markets, solid – circuit breakers, horizontal dashed – S/D ratio in representative-agent economies

Conditional Return Volatility and Risk Premium

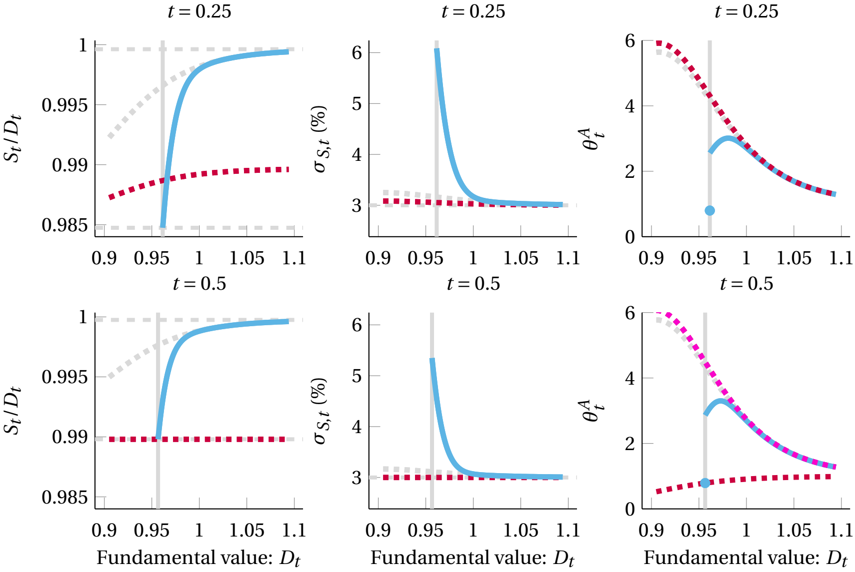


Dotted line – complete markets, solid – circuit breakers, horizontal dashed – volatility ratio in a representative-agent economy

Stronger Effects Earlier during Trading Session



Circuit breaker vs. pre-scheduled trading halt



Stochastic Disagreements

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- $\delta_0 = 0$: Agent B initially (and on average) has no biased beliefs.

- Interpretation:
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  - ↪ Investors facing leverage/risk constraint: effectively more (less) pessimistic or risk averse as the constraint tightens (loosens).

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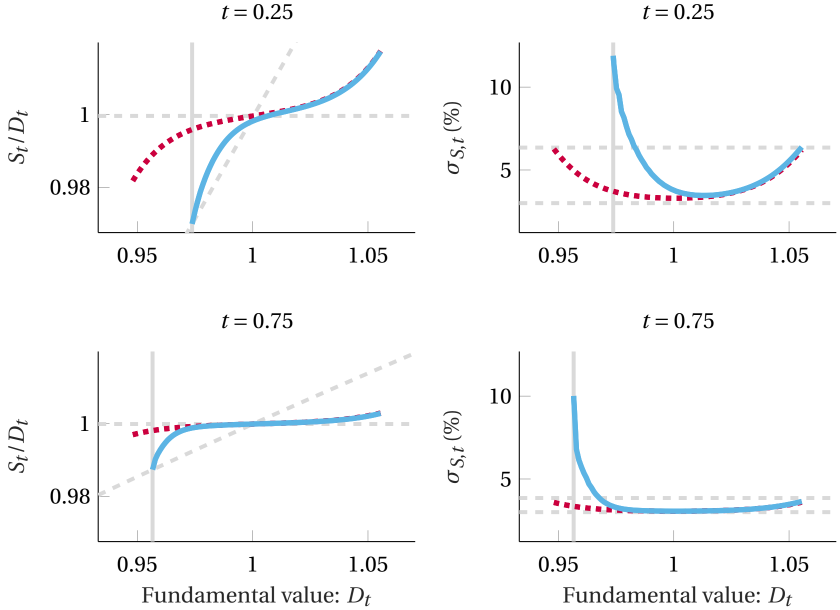
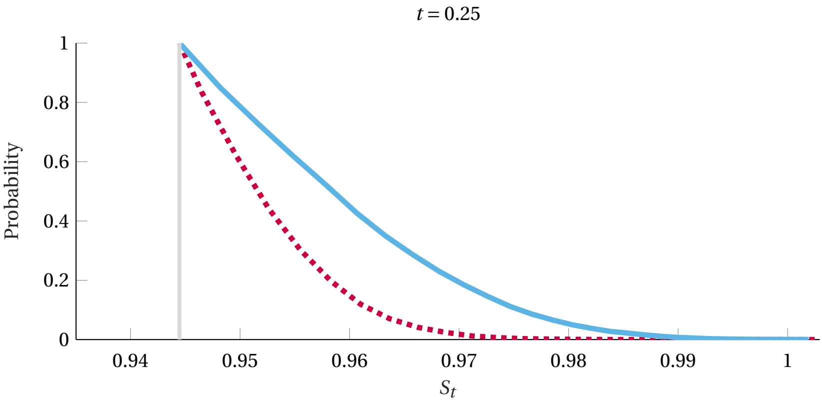
  - ↪ $\cancel{\delta = -2\%} \Rightarrow \nu = \sigma$
  - ↪ $\omega = 90\%$

CB dampen prices and amplify volatility

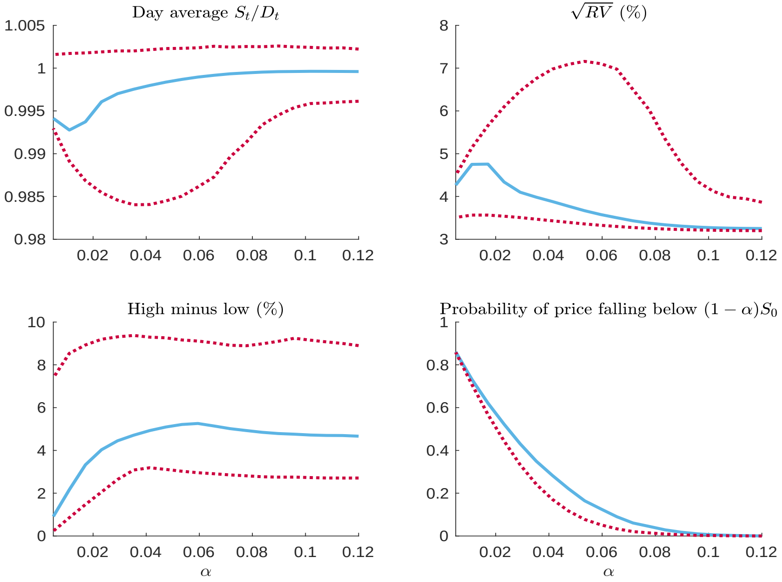


THE "MAGNET EFFECT"



- Probability to hit threshold as a function of price at $t = 0.25$
  - ↪ Dotted line: complete markets
  - ↪ Solid line: with circuit breakers

Volatility amplification and CB limit α



Welfare

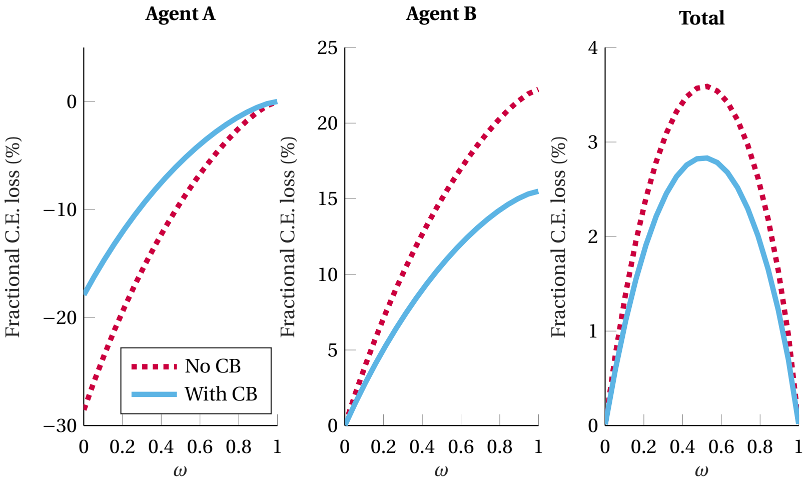
Two ways to think about welfare in this model.

- Compute welfare under agents' respective beliefs
  - $\hookrightarrow$ CBs reduce welfare
- Paternalistic view: Compute welfare under objective probability measure.
  - $\hookrightarrow$ CBs can increase welfare

Welfare Loss as a Function of Rational Agent Initial Share of Wealth

- ω – initial wealth share of agent A
- Welfare loss is relative to the complete markets case.

Welfare Loss: Paternalistic View

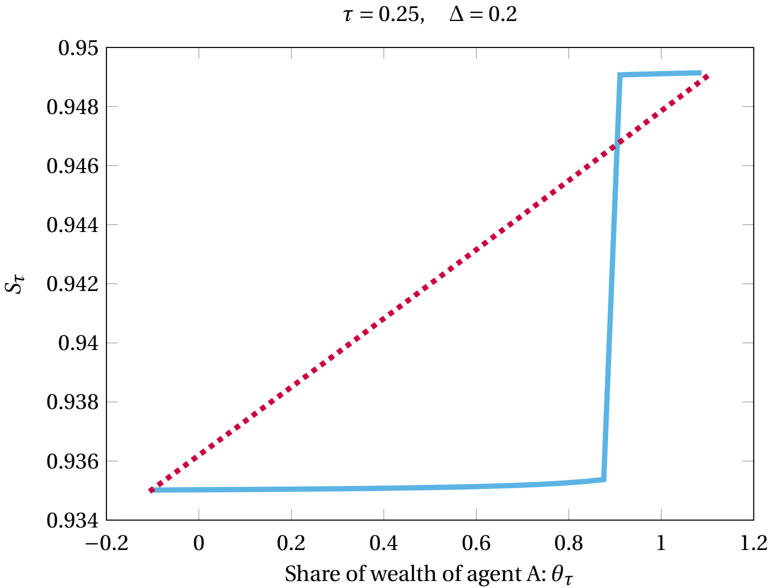


- ω – initial wealth share of agent A
- Welfare loss is relative to both agents having objective beliefs.

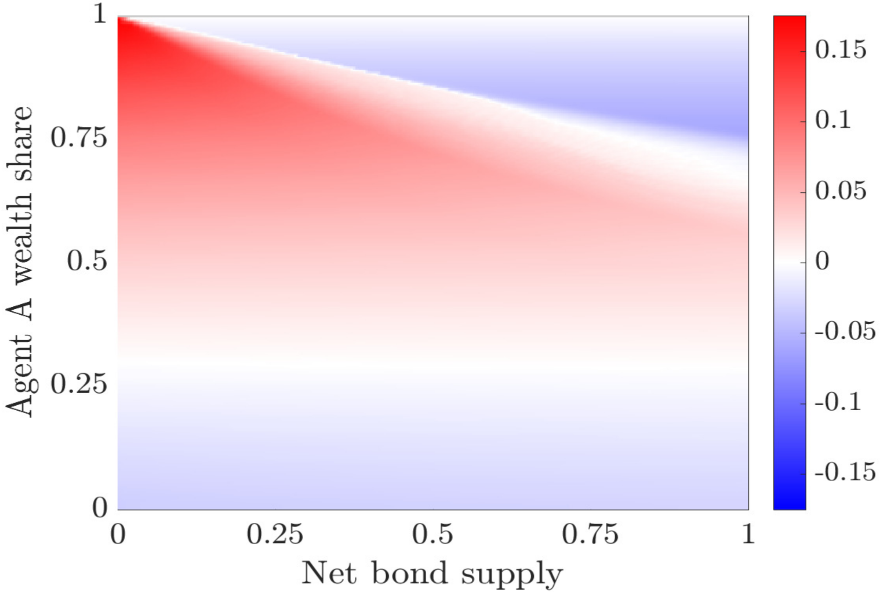
Positive Bond Supply

- With positive bond supply, it is possible that optimistic agent can hold the entire stock market at market closure.
- Could change which constraint (leverage or short-selling) becomes binding at market closing.
- If short-selling constraint binds, relative optimist becomes marginal $\Rightarrow$ Price level $\Uparrow$, volatility $\Downarrow$
- Depends on total bond supply + initial wealth distribution.

Intuition: Price upon market closure



REALIZED VOLATILITY WITH $\Delta > 0$



Robustness and Extensions

- Bounded shocks (discrete time)
  - ↪ No need to completely delever/close short positions.
  - ↪ Equilibrium can "flip" like with positive bond supply.
- Upside vs. downside CBs
- CBs based other variables: volatility, volume
- Multiple-tiered CBs

Conclusion

- A competitive benchmark to study the dynamic effects of CBs.
- CBs tend to have the following effects:
  - ↪ Lower the price-dividend ratio (increase price distortion)
  - ↪ Daily price range $\Downarrow$, conditional and realized volatility $\Uparrow$
  - ↪ Magnet Effect: raise probability of the stock price to reach the threshold limit

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- Main mechanism applies to other forms of disappearing liquidity: price limits, short-sale ban, trading frequency restrictions, sudden price jumps
- Policy implications:
  - ↪ "Reduce volatility": Which volatility?
  - ↪ Lucas critique: Danger of using historical data to estimate the likelihood of CB trigger after implementation.